Design for Lyapunov Stability and Stabilization of Control Systems

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Outline

- Review of Previous Lecture
- Design of Lyapunov Functions for Stability
  Linear Matrix Inequality (LMI)
- Design of Lyapunov Functions for Stabilization
  Bilinear Matrix Inequality (BMI)

Modern Control System Representation

State equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + bu(t)$$

Output equation

$$y(t) = cx(t)$$

Linear system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), u(t))$$

$$y(t) = g(x(t), u(t))$$

Nonlinear system

A : system matrix
b : input matrix
c : output matrix
$x(t)$: state vector
$u(t)$: input vector
$y(t)$: output vector

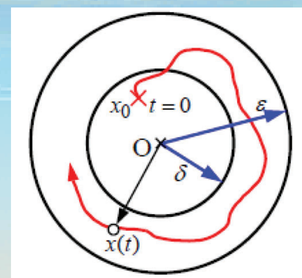
Lyapunov Stability Definitions (1)

Time-invariant autonomous (no control) system

$$\dot{x} = f(x), \quad x(0) = x_0, \ f: \text{Lipschitz C.}$$

Equilibrium point $x_e \Longleftarrow f(x_e) = 0$

Suppose $f(0) = 0 \Longrightarrow x_e = 0$



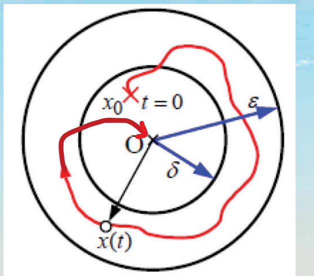
(1) The equilibrium $x_e = 0$ is stable in the sense of Lyapunov, if

$$\forall \epsilon > 0, \exists \delta > 0, \text{s.t. } \|x_0\| \le \delta \Longrightarrow \|x(t)\| \le \epsilon, \forall t \ge 0$$

Lyapunov Stability Definitions (2)

(2) The equilibrium $x_e = 0$ is asymptotically stable, if

it is stable, and $\exists \delta > 0, \text{s.t. } \|x_0\| \le \delta \Longrightarrow \lim_{t\to\infty} x(t) = 0$



Lyapunov Stable (LS)

⇑

Aymptotically Stable (AS)

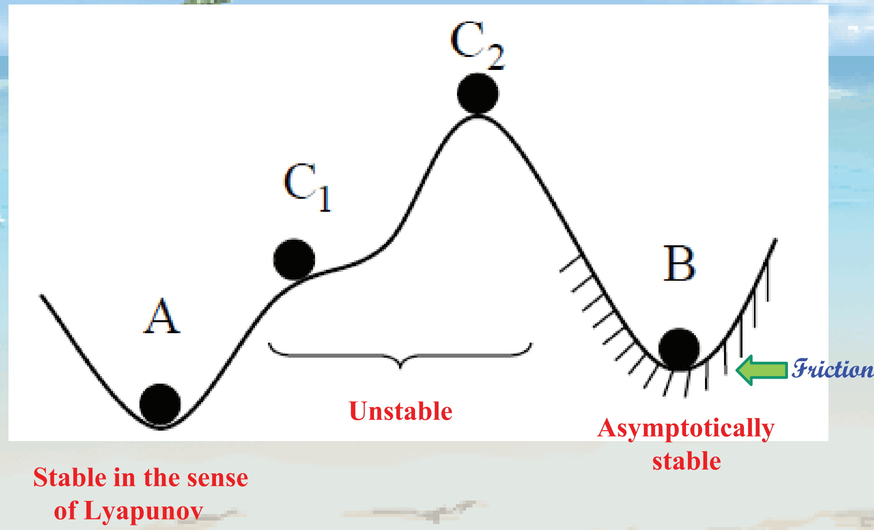
⇑

Exponentially Stable (ES)

(3) The equilibrium $x_e = 0$ is exponentially stable, if

$\exists \delta > 0, c > 0, \lambda > 0 \text{ s.t. } \|x(t)\| \le c\|x_0\| e^{-\lambda t}, \forall \|x_0\| \le \delta$

Lyapunov Stability Definitions (3)



Lyapunov Stability Theorem

If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \ne 0$$ (positive definite)

$$\dot V(x) \le 0, \quad \forall x$$ (nonpositive definite)

then $x_e = 0$ is stable (in the sense of Lyapunov)

Moreover, if

$$\dot V(x) < 0, \quad \forall x \ne 0$$ (negative definite)

then $x_e = 0$ is asymptotically stable

$V(x)$ is called a Lyapunov function (candidate)

Stability of Nonlinear Systems (1)

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2^2 \\ -x_1 x_2 - x_2 \end{bmatrix}$$

$V_3(x(t)) = x_1^2(t) + 2x_2^2(t)$

⇓

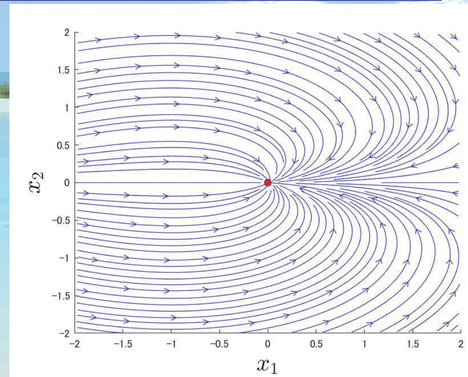
$$\frac{d}{dt}V_3 = \frac{d}{dt}\{x_1^2 + 2x_2^2\} = 2x_1\dot x_1 + 4x_2\dot x_2$$

$$= 2x_1(-x_1 + 2x_2^2) + 4x_2(-x_1x_2 - x_2)$$

$$= -2x_1^2 - 4x_2^2 = -2V_3 < 0, \quad \forall x \ne 0$$

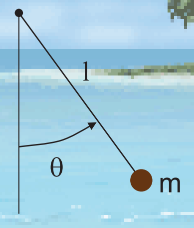
⇓

$$\lim_{t\to\infty} V_3(x(t)) = 0 \Longleftrightarrow \lim_{t\to\infty} x(t) = 0, \quad \forall x(0)$$



Aymptotically Stable (AS)

Stability of Nonlinear Systems (2)

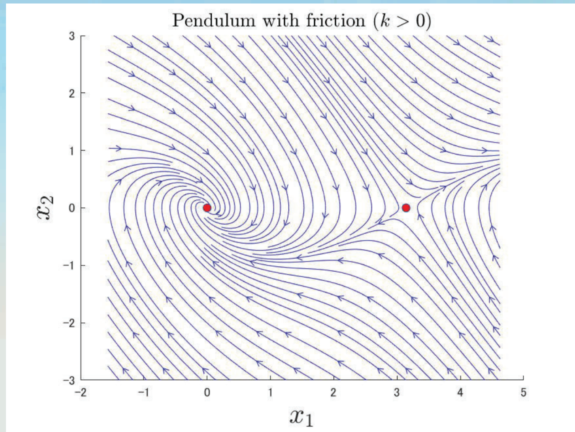


$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{mgl}{J}\sin x_1 - \frac{k}{J}x_2 \end{bmatrix}$$

$J \equiv$ moment of inertia
$k \equiv$ friction coefficient

$$J\ddot{\theta}(t) = -k\dot{\theta}(t) - mgl\sin\theta(t)$$

$$x_1 \equiv \theta\,, \quad x_2 \equiv \dot{\theta}$$



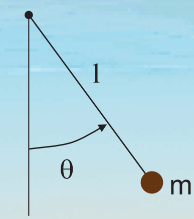
Stability of Nonlinear Systems (3)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{mgl}{J}\sin x_1 - \frac{k}{J}x_2 \end{bmatrix}$$

potential energy

$$V(x) \equiv mgl(1-\cos x_1) + \frac{1}{2}Jx_2^2$$

kinetic energy



$$\begin{aligned} \frac{d}{dt}V &= (mgl\sin x_1)\dot{x}_1 + Jx_2\dot{x}_2 \\ &= (mgl\sin x_1)x_2 + Jx_2\left(-\frac{mgl}{J}\sin x_1 - \frac{k}{J}x_2\right) \\ &= -kx_2^2 \le 0\,, \quad \forall x \end{aligned}$$

Stable in the sense of Lyapunov (actually can claim AS ...)

Design of Lyapunov Functions for Stability (1)

NO almighty method for general nonlinear systems

YES when the systems are linear $\dot{x}(t) = Ax(t)$

Consider $V(x) = x^T P x\,, \quad P > 0$

(P symmetric positive definite $\iff$ $V(x)$ positive definite)

$$\begin{aligned} \Longrightarrow \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P (Ax) \\ &= x^T \left(A^T P + PA\right) x \end{aligned}$$

($\dot{V}(x)$ negative definite $\iff$ $A^T P + PA$ symmetric negative definite)

Aymptotically Stable (AS) $\iff A^T P + PA < 0\,, \;\; P > 0$

Design of Lyapunov Functions for Stability (2)

The linear system $\dot{x}(t) = Ax(t)$ **is AS (ES) iff**

(1) All real parts of A's eigenvalues are negative

(2) $\forall Q > 0, \exists P > 0$ s.t. $A^T P + PA = -Q$ — Lyapunov Equation

(3) $\exists P > 0$ s.t. $A^T P + PA < 0$ — Linear Matrix Inequality (LMI)

(4) $\exists X > 0$ s.t. $AX + XA^T < 0$

Lyapunov equations and LMIs can be solved efficiently with MATLAB

Definition and Properties of LMI (1)

Notations

$$P > 0 \Longleftrightarrow P = P^T : \ \text{positive definite}$$
$$P < 0 \Longleftrightarrow P = P^T : \ \text{negative definite}$$

LMI formulation (in scalar variables)

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n < 0$$
$$x = [x_1, \cdots, x_n]^T, \quad F_i = F_i^T$$

LMI is convex

$$F(x) < 0, \quad F(y) < 0 \Longrightarrow$$
$$F\left(\lambda x + (1-\lambda) y\right) < 0, \quad \forall \lambda \in [0,1]$$

LMI can be simultaneous

$$F_1(x) < 0, \quad F_2(x) < 0 \Longleftrightarrow \begin{bmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{bmatrix} < 0$$

Definition and Properties of LMI (2)

LMI in matrix variable

$$P > 0, \quad A^T P + PA < 0$$

$$P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \qquad P = x_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1} + x_2 \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_2} + x_3 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_3}$$

$$\Updownarrow$$

$$P > 0 \Longleftrightarrow F_1(x) = x_1(-E_1) + x_2(-E_2) + x_3(-E_3) < 0$$

$$A^T P + PA < 0 \Longleftrightarrow$$
$$F_2(x) = x_1(A^T E_1 + E_1 A) + x_2(A^T E_2 + E_2 A) + x_3(A^T E_3 + E_3 A) < 0$$

Definition and Properties of LMI (3)

Properties of positive (negative) definite matrices

$$P > 0 \iff \forall\, \lambda(P) > 0$$
$$P < 0 \iff \forall\, \lambda(P) < 0$$
$$P > 0 \iff W^T P W > 0 \quad (|W| \neq 0)$$
$$P < 0 \iff W^T P W < 0 \quad (|W| \neq 0)$$

Schur Complement

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} > 0 \iff P_1 > 0 \quad \& \quad P_3 - P_2^T P_1^{-1} P_2 > 0$$
$$\iff P_3 > 0 \quad \& \quad P_1 - P_2 P_1^{-1} P_2^T > 0$$

Definition and Properties of LMI (4)

LMI: linear w.r.t. (matrix or scalar) variables

$$A^T P + PA < 0 \qquad XA + A^T X < 0$$

$$\begin{bmatrix} A^T P + PA & YB \\ B^T Y^T & -\tau I \end{bmatrix} < 0$$

Not LMI but equivalent to LMI

$$P^2 > I \iff \begin{bmatrix} P & I \\ I & P \end{bmatrix} > 0 \qquad P^2 < I \iff \begin{bmatrix} I & P \\ P & I \end{bmatrix} > 0$$

$$A^T P + PA + PDD^T P + E^T E < 0$$

$$\Longleftrightarrow \begin{bmatrix} A^T P + PA + E^T E & PD \\ D^T P & -I \end{bmatrix} < 0$$

Solving LMIs in Robust Control Toolbox

$$P > 0, \quad A^T P + PA < 0$$

```
A=[0 1; -2 -3];

setlmis([])
p=lmivar(1,[2 1]);

lmiterm([1 1 1 p],1,A,'s');
lmiterm([1 2 2 p],-1,1);

lmis=getlmis;
[tmin,xfeas]=feasp(lmis)

if (tmin<0)
    P=dec2mat(lmis,xfeas,p)
end
```

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} A^T P + PA & 0 \\ 0 & -P \end{bmatrix} < 0$$

↓

$$P = \begin{bmatrix} 1.5143 & 0.2958 \\ 0.2958 & 0.3484 \end{bmatrix}$$

Extension to Stability with Convergence Rate

The linear system $\dot{x}(t) = Ax(t)$

$$\|x(t)\| < c\|x(0)\|\, e^{-\mu t} \quad (\mu > 0) \quad \Longleftrightarrow$$

(1) All real parts of A's eigenvalues $< -\mu < 0$

(2) $\forall Q > 0, \exists P > 0$ s.t. $(A + \mu I)^T P + P(A + \mu I) = -Q$ — Lyapunov Equation

(3) $\exists P > 0$ s.t. $A^T P + PA < -2\mu P$

(4) $\exists X > 0$ s.t. $AX + XA^T < -2\mu X$ — Linear Matrix Inequality (LMI)

Extension to Robust Stability (1)

The uncertain linear system

$$\dot{x}(t) = Ax(t), \quad A = \lambda A_1 + (1-\lambda)A_2, \quad \lambda \in [0,1]$$

is AS for any λ if

$$\exists P > 0 \text{ s.t. } A_1^T P + PA_1 < 0, \quad A_2^T P + PA_2 < 0$$

⇕

$$\exists X > 0 \text{ s.t. } XA_1^T + A_1 X < 0, \quad XA_2^T + A_2 X < 0$$

Extension to Robust Stability (2)

The uncertain linear system

$$\dot{x}(t) = (A + DF(t)E)\,x(t), \quad \|F(t)\| \le 1$$

is AS for any $F(t)$ if

$$\exists P > 0 \text{ s.t. } (A + DF(t)E)^T P + P(A + DF(t)E) < 0$$

⇕

$$A^T P + PA + PDD^T P + E^T E < 0$$

⇕

$$\begin{bmatrix} A^T P + PA + E^T E & PD \\ D^T P & -I \end{bmatrix} < 0$$

Design of Lyapunov Functions for Stabilization

Consider state feedback for linear control system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t)$$

Closed-loop system

$$\dot{x}(t) = (A + BK)x(t)$$

Closed-loop system is AS iff

$$\exists P > 0 \ \text{ s.t. } \ (A + BK)^T P + P(A + BK) < 0$$

BMI with respect to P, K due to PBK can not be solved with MATLAB

Design of Lyapunov Functions for Stabilization

Closed-loop system

$$\dot{x}(t) = (A + BK)x(t)$$

Closed-loop system is AS iff

$$\exists X > 0 \ \text{ s.t. } \ (A + BK)X + X(A + BK)^T < 0$$

⇕ $KX = M$

$$\exists X > 0, M \ \text{ s.t. } \ AX + BM + (AX + BM)^T < 0$$

LMI with respect to X, M

⇓

$$K = MX^{-1}$$

Extension to Robust Stabilization (1)

State feedback for the uncertain linear control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t)$$
$$A = \lambda A_1 + (1 - \lambda)A_2, \quad \lambda \in [0, 1]$$

Control specification

$$\|x(t)\| < c\|x(0)\| \ e^{-\mu t} \quad (\mu > 0) \quad \forall \lambda \in [0, 1]$$

Closed-loop system

$$\dot{x}(t) = (A + BK)x(t)$$

$$\begin{aligned} A + BK &= \lambda A_1 + (1 - \lambda)A_2 + BK \\ &= \lambda(A_1 + BK) + (1 - \lambda)(A_2 + BK) \end{aligned}$$

Extension to Robust Stabilization (2)

Closed-loop system

$$\dot{x}(t) = (A + BK)x(t)$$
$$A + BK = \lambda(A_1 + BK) + (1 - \lambda)(A_2 + BK)$$

is AS and $\|x(t)\| < c\|x(0)\| \ e^{-\mu t} \quad (\mu > 0) \quad \forall \lambda \in [0, 1]$ if

⇕

$$\exists X > 0 \ \text{ s.t. } \ \begin{aligned} X(A_1 + BK)^T + (A_1 + BK)X &< -2\mu X \\ X(A_2 + BK)^T + (A_2 + BK)X &< -2\mu X \end{aligned}$$

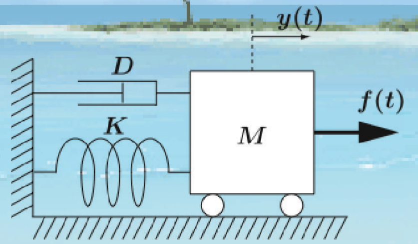
⇕ $KX = M$

LMIs

$$\begin{aligned} A_1X + BM + (A_1X + BM)^T + 2\mu X &< 0 \\ A_2X + BM + (A_2X + BM)^T + 2\mu X &< 0 \end{aligned}$$

Robust Stabilization for MDK Systems (1)

MDK (mass-spring-damper) system



Mass	$M = 1$
Spring	$K \in [2, 3]$
Damper	$D = 3$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ -K & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Robust Stabilization for MDK Systems (2)

Control Specification

$$\|x(t)\| < c\|x(0)\|\, e^{-\mu t} \quad (\mu = 3) \quad \forall K \in [2, 3]$$

$$K = 2\lambda + 3(1-\lambda), \quad \lambda \in [0, 1]$$

$$A = \lambda A_1 + (1-\lambda) A_2$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix}$$

$$X > 0 \qquad \begin{aligned} A_1 X + BM + (A_1 X + BM)^T + 2\mu X &< 0 \\ A_2 X + BM + (A_2 X + BM)^T + 2\mu X &< 0 \end{aligned}$$

Robust Stabilization for MDK Systems (3)

```
A1=[0 1; -2 -3];  A2=[0 1; -3 -3];
B=[0;1];          mu=3;

setlmis([])
x=lmivar(1,[2 1]);
m=lmivar(2,[1 2]);

lmiterm([1 1 1 x],A1+mu*eye(2),1,'s');
lmiterm([1 1 1 m],B,1,'s');
lmiterm([1 2 2 x],A2+mu*eye(2),1,'s');
lmiterm([1 2 2 m],B,1,'s');
lmiterm([1 3 3 x],-1,1);
```

Robust Control Toolbox in MATLAB

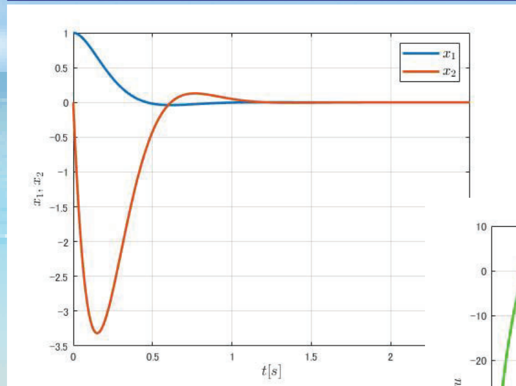
```
lmis1=getlmis;
[tmin,xfeas]=feasp(lmis1)

if (tmin<0)
    X=dec2mat(lmis1,xfeas,x)
    M=dec2mat(lmis1,xfeas,m)
    K=M*inv(X)
end
```

$$X = 10^3 * \begin{bmatrix} 0.0410 & -0.1873 \\ -0.1873 & 1.3530 \end{bmatrix}$$

$$M = [\,-675.4848 \quad -669.1301\,]$$

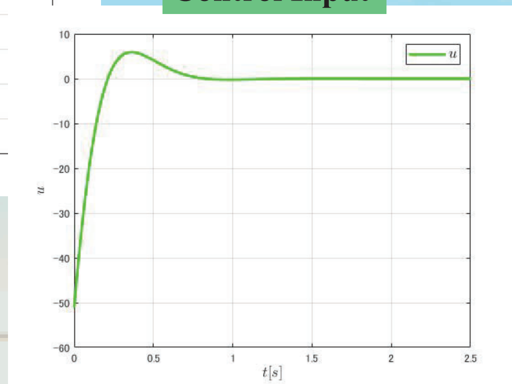
$$K = [\,-51.0492 \quad -7.5620\,]$$

Robust Stabilization for MDK Systems (4)



State Convergence

Control Input

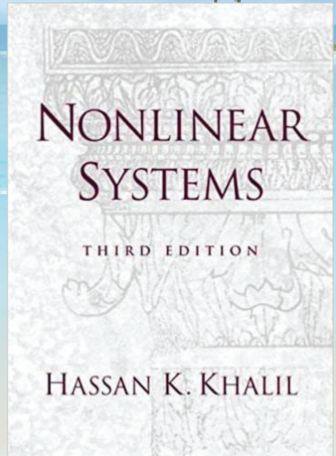


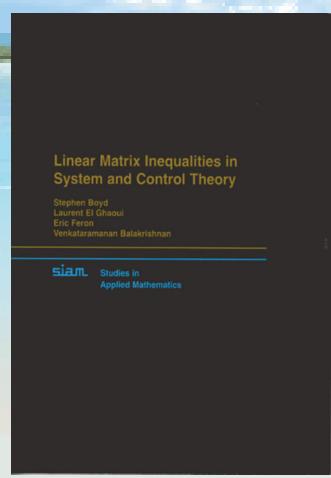
Conclusion

- Review of Lyapunov stability in modern control systems
- Definition of LMIs and properties
- LMIs for stability analysis of linear systems
- LMIs for stabilization of linear systems

- **More Topics**:
  Controller design with structure specification/limitation
  Extension to LaSalle's Invariance Principle, etc

References





(1) H. K. Khalil: Nonlinear Systems. Prentice-Hall, New Jersey, 1996.

(2) S. Boyd et al: Linear Matrix Inequalities in Systems and Control Theory, SIAM, 1994

terima kasih banyak

Thank you for your kind attention!

Additional References

- Linearized nonlinear systems

$$\dot{x}(t) = f(x(t)) \Longrightarrow \dot{x}(t) = \left(\frac{\partial f}{\partial x}\Big|_{x=0} \right) x + O(x^2)$$

- Time-variant systems

$$\dot{x}(t) = A(t)x(t)$$

- Uncertain systems (norm-bounded or polytopic)

$$\dot{x}(t) = (A + \Delta A(t, x))\, x(t)$$

$$\dot{x}(t) = \left(\sum_{i=1}^{N} \mu_i A_i \right) x(t)\,, \quad \mu_i \geq 0, \sum_{i=1}^{N} \mu_i = 1$$

- Stochastic control systems

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\, \mathrm{d}t + Hx(t)\, \mathrm{d}w(t)$$