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Lyapunov Stability Theory in Modern Control Systems

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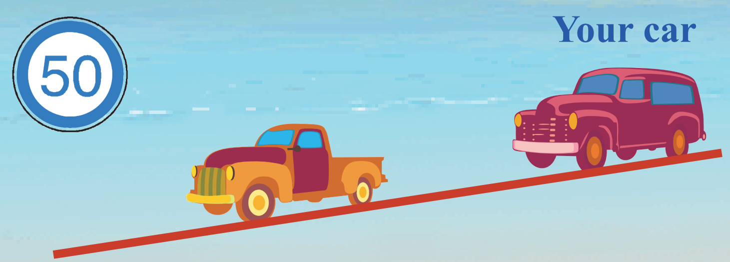
Modern Control Systems

Outline

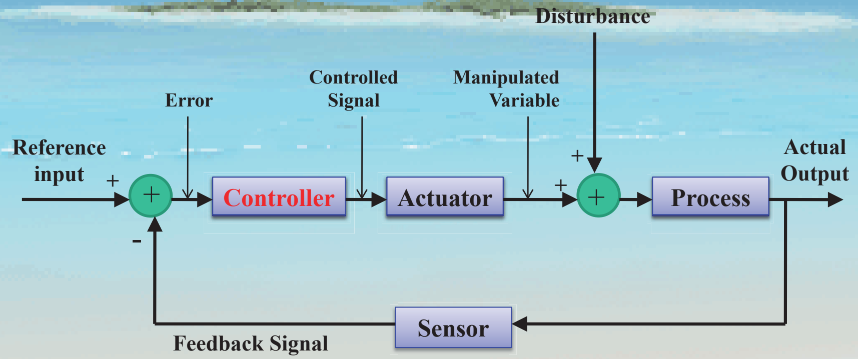
- Modern Control Systems
- Lyapunov Stability Theory
- Design of Lyapunov Functions

What is Control?



Speed (distance) control via accelerator and brake

Feedback Control System



Dynamic System

- Motion equation of MDK system

$$M\ddot{y}(t) + D\dot{y}(t) + Ky(t) = f(t)$$

$$y(t) = x_1(t) \rightarrow \dot{y}(t) = x_2(t), \quad \dot{x}_2(t) = \ddot{x}_1(t) = \ddot{y}(t)$$

$$\rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{D}{M}x_2(t) - \dfrac{K}{M}x_1(t) + \dfrac{1}{M}u(t) \end{cases}$$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{D}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} u(t)$$

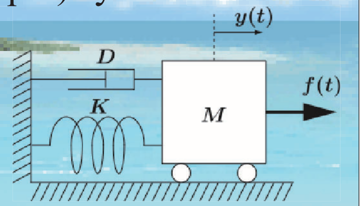
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Dynamic System

- Example 1: MDK (mass-spring-damper) system

<u>Motion equation</u>

$$M\ddot{y}(t) + D\dot{y}(t) + Ky(t) = f(t)$$



- Choose control input $f(t) = u(t)$

  to adjust displacement of mass （Output $y(t)$）

  to adjust velocity of mass （Output $\dot{y}(t)$）

  → Control the system

Dynamic System

→ State space representation (equation)

State equation
$$\frac{d x(t)}{dt} = A x(t) + b u(t)$$

Output equation
$$y(t) = c x(t)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$: state (variable) vector

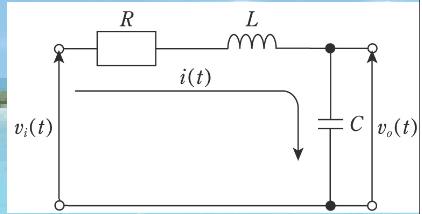
A : system matrix

b : input matrix

c : output matrix

Dynamic System

- Example 2: LCR circuit



<u>Voltage equation</u>

$$L\dot{i}(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v_i(t)$$

- Choose control input $v_i(t) = u(t)$ to adjust the voltage on C (Output $y(t)$)

→ Control (regulate) the system

Dynamic System

→ State space representation (linear system)

State equation
$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$$

Output equation
$$y(t) = \boldsymbol{c}\boldsymbol{x}(t)$$

$\boldsymbol{x}(t)$: state (variable) vector

$\boldsymbol{A}$: system matrix
$\boldsymbol{b}$: input matrix
$\boldsymbol{c}$: output matrix

Dynamic System

- Voltage equation of LCR circuit

$$L\dot{i}(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v_i(t)$$

↓

$$x_1(t) = i(t), \quad x_2(t) = \frac{1}{C}\int_0^t i(\tau)\, d\tau$$

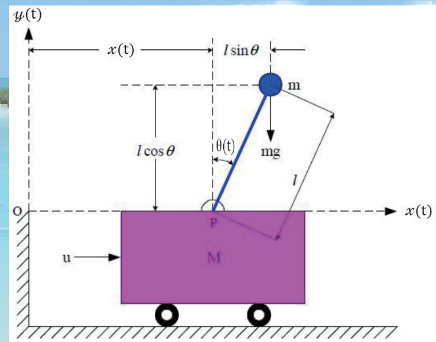
↓

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Dynamic System

- Example 3:
Inverted pendulum on a cart



<u>Motion equation</u>

$$m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta$$

$$(M+m)\ddot{x} - ml\sin\theta\dot{\theta}^2 + ml\cos\theta\ddot{\theta} = u$$

- Choose control input $f(t) = u(t)$ to adjust angle of pendulum (Output $y(t) = \theta(t)$)

Dynamic System

$$\begin{cases} \ddot{x} = \dfrac{u + ml\sin\theta\dot{\theta}^2 - mg\cos\theta\sin\theta}{M + m - m\cos^2\theta} \\[2ex] \ddot{\theta} = \dfrac{u\cos\theta - (M+m)g\sin\theta + ml\cos\theta\sin\theta\dot{\theta}^2}{ml\cos^2\theta - (M+m)l} \end{cases}$$

$$x_1 = \theta \qquad x_2 = \dot{\theta} = \dot{x_1} \qquad x_3 = x \qquad x_4 = \dot{x} = \dot{x_3}$$

$$\begin{cases} \dot{x_1} = x_2 \\[1ex] \dot{x_2} = \dfrac{u\cos x_1 - (M+m)g\sin x_1 + ml\cos x_1\sin x_1 \dot{x_2}^2}{ml\cos^2 x_1 - (M+m)l} \\[2ex] \dot{x_3} = x_4 \\[1ex] \dot{x_4} = \dfrac{u + ml\sin x_1 x_2{}^2 - mg\cos x_1\sin x_1}{M + m - m\cos^2 x_1} \end{cases}$$

Dynamic System

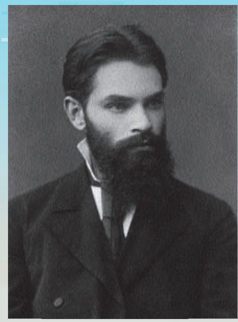
State space representation (nonlinear system)

State equation $\qquad \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), u(t))$

Output equation $\qquad y(t) = g(x(t), u(t))$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$: state (variable) vector

Lyapunov Stability Theory



Lyapunov
(1857-1918)

Stability of Linear Systems

$$\dot{x}(t) = Ax(t) \quad \rightarrow \quad x(t) = e^{At}x(0)$$

e^{At} : combination of $e^{\lambda_i(A)t}$

Aymptotically Stable (**AS**) if and only if

$$\lim_{t\to\infty} x(t) = 0, \quad \forall x(0)$$

⇕

The real parts of eigenvalues of A are all **negative**

Stability of Linear Systems

$\dot{x}(t) = Ax(t)$ Aymptotically Stable (AS)

⇕

$$\lim_{t\to\infty} x(t) = 0\,, \quad \forall x(0)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

⇕

$$\lim_{t\to\infty} \left[x_1^2(t) + x_2^2(t) + \cdots + x_n^2(t)\right] = 0$$

Stability of Linear Systems

$$\lim_{t\to\infty} \frac{d}{dt} V_1(x(t)) < 0\,, \quad \forall t\,, \forall x(t) \neq 0$$

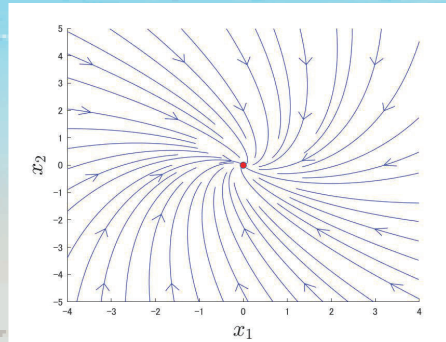
works well for some systems

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix} x$$

$$\frac{d}{dt}V_1 = \frac{d}{dt}\{x_1^2 + x_2^2\} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1(-2x_1 + x_2) + 2x_2(-x_1 - 3x_2)$$

$$= -4x_1^2 - 6x_2^2 < 0\,, \quad \forall x \neq 0$$



Stability of Linear Systems

How to check this limit for any initial value?

$$\lim_{t\to\infty} \underbrace{\left[x_1^2(t) + x_2^2(t) + \cdots + x_n^2(t)\right]}_{V_1(x(t))} = 0$$

⇑

$$\lim_{t\to\infty} \frac{d}{dt} V_1(x(t)) < 0\,, \quad \forall t\,, \forall x(t) \neq 0$$

Stability of Linear Systems

$$\lim_{t\to\infty} \frac{d}{dt} V_1(x(t)) < 0\,, \quad \forall t\,, \forall x(t) \neq 0$$

NOT work well for some other systems

$$\dot{x} = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix} x$$

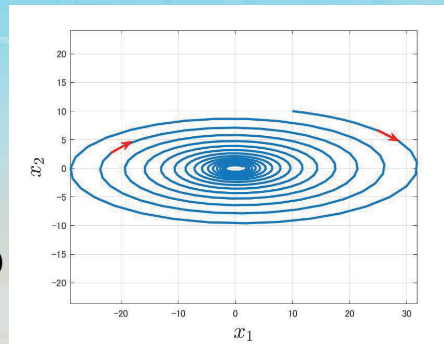
$$\frac{d}{dt}V_1 = \frac{d}{dt}\{x_1^2 + x_2^2\} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1(-x_1 + 100x_2) + 2x_2(-10x_1 - x_2)$$

$$= -2x_1^2 + 180x_1x_2 - 2x_2^2$$

$$\not< 0\,, \quad \forall x \neq 0$$

?



Stability of Linear Systems

$$\dot{x} = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix} x$$

$$V_1(x(t)) = x_1^2(t) + x_2^2(t) \Longrightarrow V_2(x(t)) = x_1^2(t) + 10x_2^2(t)$$

$$\frac{d}{dt}V_2 = \frac{d}{dt}\{x_1^2 + 10x_2^2\} = 2x_1\dot{x}_1 + 20x_2\dot{x}_2$$

$$= 2x_1(-x_1 + 100x_2) + 20x_2(-10x_1 - x_2)$$

$$= -2x_1^2 - 20x_2^2 < 0\,, \quad \forall x \neq 0$$

$$\lim_{t\to\infty} V_2(x(t)) = 0 \Longleftrightarrow \lim_{t\to\infty} x(t) = 0\,, \quad \forall x(0)$$

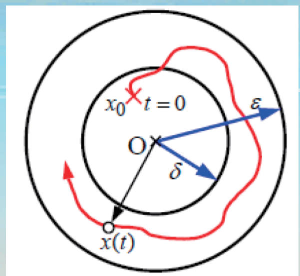
Lyapunov Stability Definitions

Time-invariant autonomous (no control) system

$$\dot{x} = f(x)\,, \quad x(0) = x_0\,, \ f: \text{Lipschitz C.}$$

Equilibrium point $x_e \Longleftarrow f(x_e) = 0$

Suppose $f(0) = 0 \Longrightarrow x_e = 0$



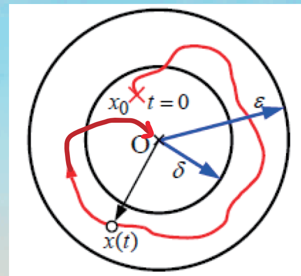
(1) The equilibrium $x_e = 0$ is stable in the sense of Lyapunov, if

$$\forall \epsilon > 0, \exists \delta > 0, \text{s.t.} \ \|x_0\| \leq \delta \Longrightarrow \|x(t)\| \leq \epsilon, \forall t \geq 0$$

Lyapunov Stability Definitions

(2) The equilibrium $x_e = 0$ is asymptotically stable, if

it is stable, and $\exists \delta > 0, \text{s.t.} \ \|x_0\| \leq \delta \Longrightarrow \lim_{t\to\infty} x(t) = 0$



Lyapunov Stable (LS)

↑

Aymptotically Stable (AS)

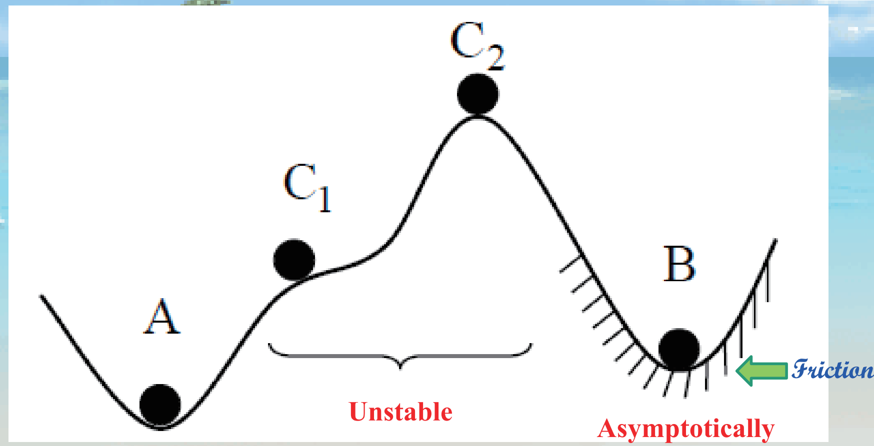
↑

Exponentially Stable (ES)

(3) The equilibrium $x_e = 0$ is exponentially stable, if

$$\exists \delta > 0, c > 0, \lambda > 0 \ \text{s.t.} \ \|x(t)\| \leq c\|x_0\|e^{-\lambda t}, \forall \|x_0\| \leq \delta$$

Lyapunov Stability Definitions

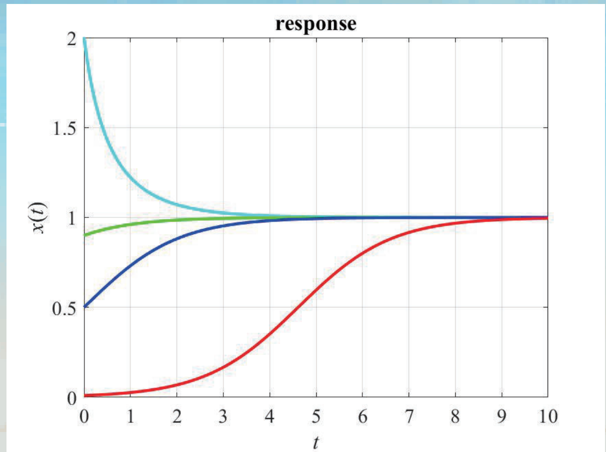


Stable in the sense of Lyapunov

Asymptotically stable

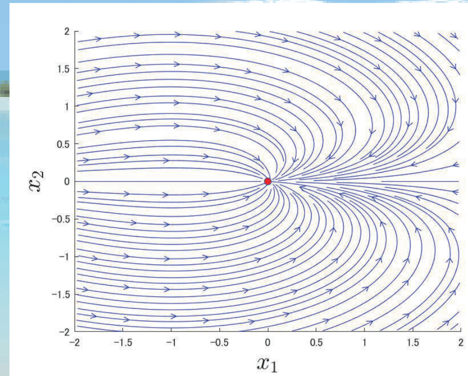
Coffee Break

$$\dot{x} = x(1-x), \quad x(0) = 0.01, 0.5, 0.9, 2.0$$



Stability of Nonlinear Systems

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2^2 \\ -x_1 x_2 - x_2 \end{bmatrix}$$



$$V_3(x(t)) = x_1^2(t) + 2x_2^2(t)$$

$$\frac{d}{dt}V_3 = \frac{d}{dt}\{x_1^2 + 2x_2^2\} = 2x_1\dot{x}_1 + 4x_2\dot{x}_2$$

$$= 2x_1(-x_1 + 2x_2^2) + 4x_2(-x_1x_2 - x_2)$$

$$= -2x_1^2 - 4x_2^2 = -2V_3 < 0, \quad \forall x \neq 0$$

Aymptotically Stable (AS)

$$\lim_{t\to\infty} V_3(x(t)) = 0 \Longleftrightarrow \lim_{t\to\infty} x(t) = 0, \quad \forall x(0)$$

Lyapunov Stability Theorem

If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \neq 0$$ (positive definite)

$$\dot{V}(x) \leq 0, \quad \forall x$$ (nonpositive definite)

then $x_e = 0$ is stable (in the sense of Lyapunov)

Moreover, if

$$\dot{V}(x) < 0, \quad \forall x \neq 0$$ (negative definite)

then $x_e = 0$ is asymptotically stable

$V(x)$ is called a Lyapunov function (candidate)

Conclusion & References

- Representation of modern control systems
- Definition of various system stability
- Basic Lyapunov stability theory

- **Next Lecture**:
  Further analysis and design of Lyapunov Theory
  Extension to LaSalle's Invariance Principle, etc

Hassan K. Khalil: Nonlinear Systems. Prentice-Hall, New Jersey, 1996.