Theory and Applications of Algebraic Hyperstructures

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Contents

- Two generalizations of groups
- Connections between algebraic hyperstructures and ordinary algebra by using the concept of fundamental relation
- Part 3: Fuzzy sets and their generalizations
- Part 4: Rough sets

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Group

A system $(G, \cdot)$, where G is a non-empty set and $\cdot$ is a binary operation, is called a *group* if it satisfies the following conditions:

(1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in G$,

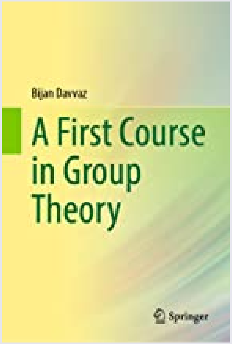
(2) the equations $a \cdot x = b$ and $y \cdot a = b$ have solutions in G, for all $a, b, c \in G$.

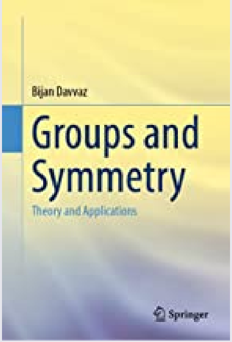
$f : G \times G \to G$

(1) $f\big(a, f(b, c)\big) = f\big(f(a, b), c\big)$, for all $a, b, c \in G$,
(2) the equations $f(a, x) = b$ and $f(y, a) = b$ have solutions in G, for all $a, b, c \in G$.

Group

Some books related to groups





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The first generalization

The non-empty set G together with an n-ary operation

$$f : G^n \longrightarrow G$$

is called an *n-ary groupoid* and is denoted by (G, f).
According to the general convention used in the theory of such groupoids the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. The notion of an n-ary group is a natural generalization of the notion of a group and has many applications in different branches.

The first generalization

A non-empty set G with a ternary operation $f : G \times G \times G \to G$ is called a *ternary group* if

(1) For every $x_1, x_2, \ldots, x_n \in G$,

$$\begin{aligned} f\Big(f(x_1, x_2, x_3), x_4, x_5\Big) &= f\Big(x_1, f(x_2, x_3, x_4), x_5\Big) \\ &= f\Big(x_1, x_2, f(x_3, x_4, x_5)\Big), \end{aligned}$$

(2) For all $b, a_1, a_2, a_3 \in G$ the following equations have solutions

$$f(x, a_1, a_2) = b, \quad f(a_1, y, a_3) = b \ \text{ and } \ f(z, a_1, a_2) = b.$$

The first generalization

In general, we have the following definition.

Definition

A non-empty set G with an n-ary operation f is called an *n-ary group* if

(1) For every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in G$,

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

(2) for all $b, a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n \in G$ $(k = 1, \ldots, n)$ there exists a unique $z \in G$ such that

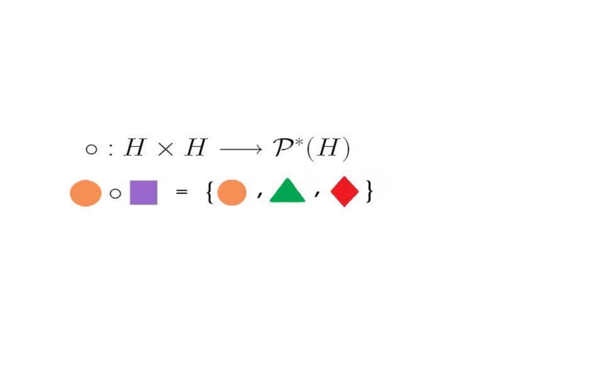
$$f(a_1^{k-1}, z, a_{k+1}^n) = b.$$

The first generalization

If $(G, \cdot)$ is a group, then (G, f) is an n-ary group, where $f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$. But for every $n \geq 3$ there are n-ary groups which are not derived from any group.

The second generalization

Let H be a non-empty set and $\star : H \times H \longrightarrow \mathcal{P}(H) \setminus \emptyset$ be a *hyperoperation*. The couple $(H, \star)$ is called a *hypergroupoid*.



Blood Groups :

$\otimes$	O	A	B	AB
O	O	O A	O B	A B
A	O A	O A	AB A B O	AB A B
B	O B	AB A B O	O B	AB A B
AB	A B	AB A B	AB A B	AB A B

The second generalization

Definition

A hypergroupoid $(H, \star)$ is called a *hypergroup* if

(1) $a \star (b \star c) = (a \star b) \star c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in a \star b} u \star c = \bigcup_{v \in b \star c} a \star v.$$

(2) $a \star H = H \star a = H$ for all $a \in H$. This condition is called the *reproduction axiom*.

The second condition is frequently used in the form: Given $a, b \in H$, there exist $x, y \in H$ such that $b \in a \star x$ and $b \in y \star a$.

The second generalization

Definition

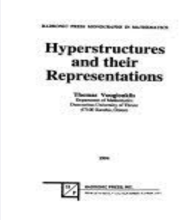
A multi-valued system $\mathcal{M} =< P, \star, e, ^{-1} >$ where $e \in P$, $^{-1} : P \longrightarrow P$, $\star : P \times P \rightarrow \mathcal{P}(P) \setminus \emptyset$ is called a *polygroup* if the following axioms hold for all $x, y, z \in P$:

(1) $(x \star y) \star z = x \star (y \star z)$,

(2) $x \star e = x = e \star x$,

(3) $x \in y \star z$ implies $y \in x \star z^{-1}$ and $z \in y^{-1} \star x$.

The second generalization



Weak hyperstructures (or H_v-structures) first introduced by Vougiouklis in Fourth AHA congress (1990).

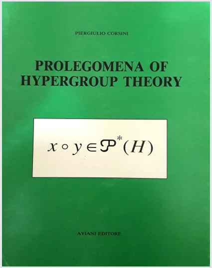
Definition

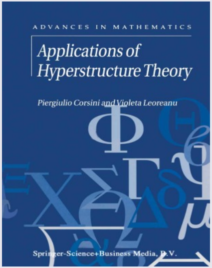
A hyperstructure $(H, \star)$ is called an H_v-group if

(1) $x \star (y \star z) \cap (x \star y) \star z \neq \emptyset$ for all $x, y, z \in H$,

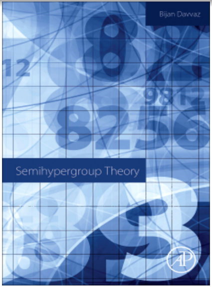
(2) $a \star H = H \star a = H$ for all $a \in H$.

Some books related to hypergroups





Corsini **Leoreanu-Fotea**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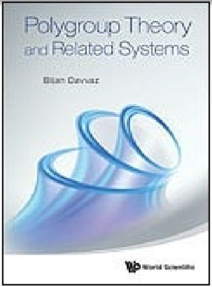
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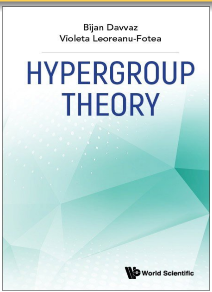
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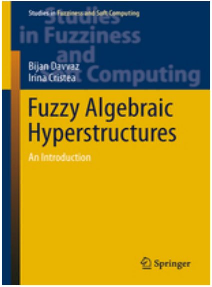
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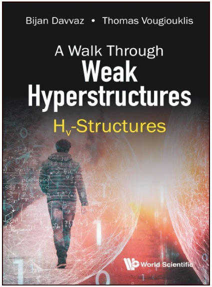
Springer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A Walk Through
Weak Hyperstructures
H_v-Structures
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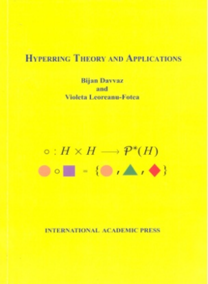
Hyperring Theory and Applications
Bijan Davvaz
and
Violeta Leoreanu-Fotea
$\circ : H \times H \longrightarrow \mathcal{P}^*(H)$
International Academic Press

Some references related to Chemistry, Biology and Physics

Mathematical Biosciences 285 (2017) 112–118

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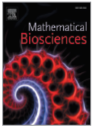

Algebraic hyperstructures associated to biological inheritance

CrossMark

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[b] Department of Mathematics, Yazd University, Yazd, Iran

Some references related to Chemistry, Biology and Physics

B. Davvaz, A. Dehghan Nezad and M. Mazloum-Ardakani, *Chemical hyperalgebra: Redox reactions*, MATCH Communications in Mathematical and in Computer Chemistry, 71 (2014) 323-331.

B. Davvaz, A. Dehghan Nezad, M. Mazloum-Ardakani and M. A. Sheikh-Mohseneib, *Describing the algebraic hyperstructure of all elements in radiolytic processes in cement medium*, MATCH Commun. Math. Comput. Chem., 72(2) (2014) 375-388.

Some references related to Chemistry, Biology and Physics

- B. Davvaz, *Weak algebraic hyperstructures as a model for interpretation of chemical reactions*, Iranian Journal of Mathematical Chemistry, 7(2) (2016) 267-283.
- M. Al Tahan and B. Davvaz, *Electrochemical cells as experimental verifications of n-ary hyperstructures*, Matematika, 35(1) (2019) 13-24 .
- M. Ghadiri, B. Davvaz and R. Nekouian, *H_v-Semigroup structure on F_2-offspring of a gene pool*, International Journal of Biomathematics, 5(4) (2012) 1250011 (13 pages).

Some references related to Chemistry, Biology and Physics

- B. Davvaz, A. Dehghan Nezad and M. M. Heidari, *Inheritance examples of algebraic hyperstructures*, Information Sciences, 224 (2013) 180-187.
- M. Al Tahan and B. Davvaz, *Hyperstructures associated to Biological inheritance*, Mathematical Biosciences, 285 (2017) 112-118.
- B. Davvaz, A Dehghan Nezhad and S. M. Moosavi Nejad, *Algebraic hyperstructure of observable elementary particles including the Higgs boson*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences,

Regular relations

By using a certain type of equivalence relations, we can connect semihypergroups to semigroups and hypergroups to groups. These equivalence relations are called strongly regular relations. More exactly, starting with a (semi)hypergroup and using a strongly regular relation, we can construct a (semi)group structure on the quotient set. A natural question arises: Do they also exist regular relations? The answer is positive, regular relations provide us new (semi)hypergroup structures on the quotient sets.

Regular relations

Let $(H, \star)$ be a semihypergroup and ρ be an equivalence relation on H. If A and B are non-empty subsets of H, then $A\overline{\rho}B$ means that for all $a \in A$ there exists $b \in B$ such that $a\rho b$ and for all $b' \in B$ there exists $a' \in A$ such that $a'\rho b'$; also $A\overline{\overline{\rho}}B$ means that for all $a \in A$ and for all $b \in B$, we have $a\rho b$.

Regular relations

The equivalence relation ρ is called

(1) regular on the right (on the left) if for all x of H, from $a\rho b$, it follows that $(a\star x)\overline{\rho}(b\star x)$ $((x\star a)\overline{\rho}(x\star b)$ respectively);

(2) strongly regular on the right (on the left) if for all x of H, from $a\rho b$, it follows that $(a\star x)\overline{\overline{\rho}}(b\star x)$ $((x\star a)\overline{\overline{\rho}}(x\star b)$ respectively);

(3) ρ is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left.

Regular relations

Theorem

Let $(H, \star)$ be a (semi)hypergroup and ρ be an equivalence relation on H.

(1) *If ρ is regular, then H/ρ is a (semi)hypergroup with respect to the following hyperoperation:*
$\rho(x) \odot \rho(y) = \{\rho(z) \mid z \in x \star y\}$;

(2) *If the above hyperoperation is well defined on H/ρ, then ρ is regular.*

Regular relations

Theorem

Let $(H, \star)$ be a (semi)hypergroup and ρ be an equivalence relation on H.

(1) *If ρ is strongly regular, then H/ρ is a (semi)group with respect to the following operation:*
$\rho(x) \odot \rho(y) = \rho(z), \text{ for all } z \in x \star y;$

(2) *If the above operation is well defined on H/ρ, then ρ is strongly regular.*

Regular relations

So, by strongly regular relations we obtain a group from a hypergroup (see the following figure).

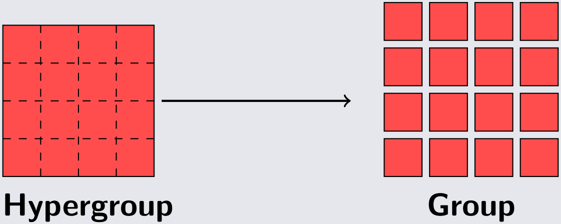


Figura: A strongly regular relation gives us a group

Fundamental relations

For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \star)$ as follows:

$$x\beta_n y \text{ if there exist } a_1, \ldots, a_n \text{ in } H \text{ such that} \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H.

Fundamental relations

This relation was introduced by Koskas and studied mainly by Corsini, Davvaz, Freni, Vougiouklis, and many others. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β. Freni proved that $\beta = \beta^*$ for hypergroups.

Theorem

If $(H, \star)$ is a hypergroup, then the relation β^* is the smallest equivalence relation on H such that the quotient H/β^* is a group. **This group is called the fundamental group**.

Open Problem:

Is it true $\beta = \beta^*$ for H_v-groups?

Fundamental relations

There exists semihypergroups in which the relation β is not transitive.

Example

Let $H = \{a, \ b, \ c, \ d\}$ be a semihypergroup with the following hyperoperation.

$\circ$	a	b	c	d
a	$\{b, c\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$

Then, it is easy to see that $c\beta^* d$ but not $c\beta d$.

Certain relations

We begin with the following question.

Whether there are any strongly regular relations on a hypergroup to obtain an abelian group, a cyclic group, a solvable group, or a nilpotent group?

$$\textbf{Hypergroup} \longrightarrow \begin{cases} \textbf{Abelian group} \\ \textbf{Cyclic group} \\ \textbf{Solvable group} \\ \textbf{Nilpotent group} \end{cases}$$

Certain relations

Freni introduced the relation Γ as a generalization of the relation β.

D. Freni, *A new characterization of the derived hypergroup via strongly regular equivalences*, Comm. Algebra, 30(8) (2002) 3977–3989.

Certain relations

Let H be a (semi)hypergroup. Then, we set $\Gamma_1 = \{(x,x) \mid x \in H\}$ and for every integer $n > 1$, Γ_n is the relation defined as follows:

$$x \Gamma_n y$$
$$\Leftrightarrow$$
$$\exists (a_1, \ldots, a_n) \in H^n,\ \exists \sigma \in \mathbb{S}_n \ :\ x \in \prod_{i=1}^{n} a_i,\ y \in \prod_{i=1}^{n} a_{\sigma(i)},$$

where $\mathbb{S}_n$ is the symmetric group on n letters.

Certain relations

The relations Γ_n are symmetric, and the relation $\Gamma = \bigcup_{n\geq 1} \Gamma_n$ is reflexive and symmetric. Let Γ^* be the transitive closure of Γ. The relation Γ^* is a strongly regular relation. The quotient H/Γ^* is a commutative semigroup. Furthermore,

Theorem

If H is a hypergroup, then H/Γ^* is an abelian group (Freni).

Certain relations

The relation Γ^* is the smallest strongly regular relation on a semihypergroup H such that the quotient H/Γ^* is a commutative semigroup. Similar to the fundamental relation β, we have the following result.

Theorem

If $(H, \star)$ is a hypergroup, then $\Gamma = \Gamma^*$ (Freni).

Certain relations

Mousavi, Leoreanu-Fotea and Jafarpour introduced a strongly regular relation ρ_A^* on the hypergroup H, such that in a particular case the quotient H/ρ_A^* is a cyclic group.

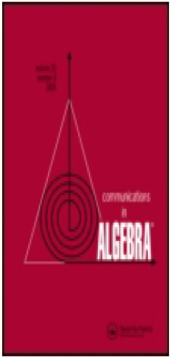
S. Ah. Mousavi, V. Leoreanu-Fotea and M. Jafarpour, *Cyclic groups obtained as quotient hypergroups*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), 61(1) (2015) 109–122.

Certain relations

Aghabozorghi

Jafarpor

Aghabozorgi, Davvaz and Jafarpour introduced and analyzed a new strongly regular relation on a hypergroup H, is named τ_n^*, such that H/τ_n^* is a solvable group.



Communications in Algebra

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Solvable Polygroups and Derived Subpolygroups

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Solvable groups derived from hypergroups

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Certain relations

Indeed, for any hypergroup H and any natural number n, we define $H^{(0)} = H$ and

$$H^{(k+1)} = \left\{ h \in H^{(k)} \mid x \star y \cap h \star y \star x \neq \emptyset \text{ for some } x, y \in H^{(k)} \right\},$$

for all $k \geq 0$. Suppose that $n \in \mathbb{N}$ and $\tau_n = \bigcup_{m>1} \tau_{m,n}$, where $\tau_{1,n}$ is the diagonal relation and for every integer $m > 1$, $\tau_{m,n}$ is the relation defined as follows:

$$x \tau_{m,n} y \Leftrightarrow \exists (z_1, \ldots, z_m) \in H^m, \exists \sigma \in \mathbb{S}_m :$$
$$\sigma(i) = i \ \text{ if } z_i \notin H^{(n)} \text{ such that } x \in \prod_{i=1}^{m} z_i \text{ and } y \in \prod_{i=1}^{m} z_{\sigma(i)}.$$

Certain relations

Obviously, for every $n \geq 1$, the relation τ_n is reflexive and symmetric. Now, let τ_n^* be the transitive closure of τ_n.

Theorem

For every $n \in \mathbb{N}$, the relation τ_n^* is a strongly regular relation.

Theorem

H/τ^* is a solvable group.

Certain relations

Also, Aghabozorgi, Davvaz and Jafarpour introduced another strongly regular relation ν^* on a hypergroup H such that the quotient H/ν^*, the set of all equivalence classes, is a nilpotent group.

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Nilpotent groups derived from hypergroups

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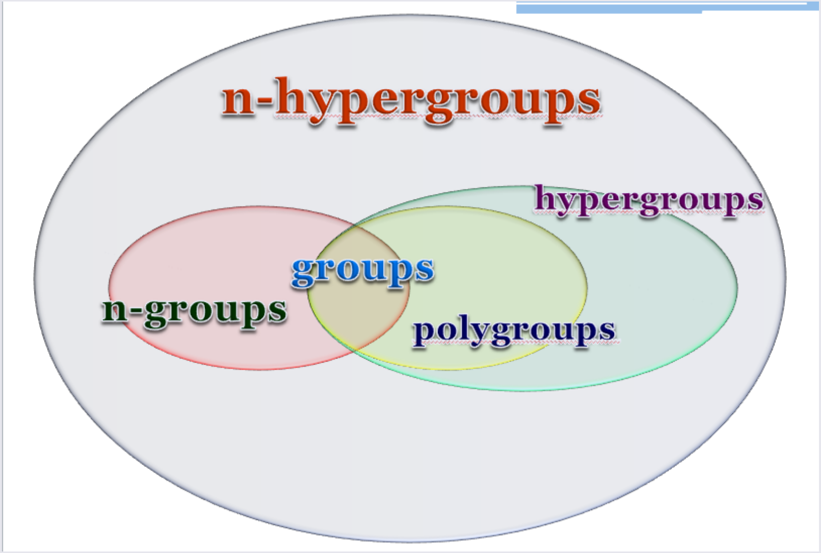
Certain relations

Recently, Shirvani, Mirvakili and Davvaz determined a family $\mathfrak{P}_{\mathfrak{R}}^{H}$ of subsets of a hypergroup H such that the geometric space $(H, \mathfrak{P}_{\mathfrak{R}}^{H})$ is strongly transitive.

M. Shirvani, S. Mirvakili and B. Davvaz, *Strongly transitive geometric spaces on hypergroups from strongly U-regular relations*, *J. Algebra Appl.*, (2021) doi.org/10.1142/S0219498822500943.


n-hypergroups
hypergroups
groups
n-groups
polygroups

Fuzzy sets

Fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets have been introduced by L. A. Zadeh (1965) as an extension of the classical notion of set.

Fuzzy sets

Definition

Let X be a set. A *fuzzy subset* A of X is characterized by a membership function $\mu_A : X \to [0,1]$ which associates with each point $x \in X$ its *grade* or *degree of membership* $\mu_A(x)$.

Example

We can define a possible membership function for the fuzzy subsets of real numbers close to zero as follows:

$$\mu_A(x) = \frac{1}{1+10x^2}.$$

Fuzzy sets

Definition

Let A and B be fuzzy subsets of X.

- $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$.
- $A = B$ if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$.
- $C = A \cup B$ if and only if $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.
- $D = A \cap B$ if and only if $\mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.

The *complement* of A, denoted by A^c, is defined by

$$\mu_{A^c}(x) = 1 - \mu_A(x), \text{ for all } x \in X.$$

Fuzzy sets

Definition

Let f be a mapping from a set X to a set Y. Let μ be a fuzzy subset of X and λ be a fuzzy subset of Y. Then, the *inverse image* $f^{-1}(\lambda)$ of λ is the fuzzy subset of X defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$, for all $x \in X$. The *image* $f(\mu)$ of μ is the fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(t) \mid t \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$.

Fuzzy sets and hyperstructures

In 1971, Rosenfeld introduced the fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. There is a considerable amount of work on the association between fuzzy sets and hyperstructures. This work can be classified into three groups. A first group of works studies crisp hyperoperations defined through fuzzy sets. This study was initiated by Corsini and others.

Fuzzy sets and hyperstructures

A second group of works concerns the fuzzy hyperalgebras. This is a direct extension of the concept of fuzzy algebras. This idea was applied by Zahedi and his group on polygroups. A third group deals also with fuzzy hyperstructures, but with a completely different approach. This was studied by Corsini, Zahedi and others. The basic idea is the following one: a crisp hyperoperation assigns to every pair of elements a crisp set; a fuzzy hyperoperation assigns to every pair of elements a fuzzy set.

Fuzzy sets and hyperstructures

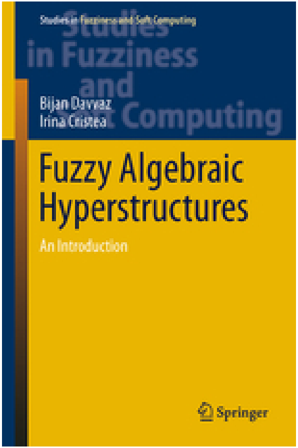
In 1999, I introduced the notion of fuzzy subhypergroup of a hypergroup.
Let $(H, \cdot)$ be a hypergroup and let μ be a fuzzy subset of H. Then, μ is said to be a *fuzzy subhypergroup* of H if the following axioms hold.

(1) $\min\{\mu(x), \mu(y)\} \leq \inf_{\alpha \in x \cdot y} \{\mu(\alpha)\}$, for all $x, y \in H$;

(2) For all $x, a \in H$ there exists $y \in H$ such that $x \in a \cdot y$ and $\min\{\mu(a), \mu(x)\} \leq \mu(y)$;

(3) For all $x, a \in H$ there exists $z \in H$ such that $x \in z \cdot a$ and $\min\{\mu(a), \mu(x)\} \leq \mu(z)$.

Fuzzy sets and hyperstructures



Intuitionistic fuzzy sets

As an important generalization of the notion of fuzzy sets on X, Atanassov introduced the concept of *intuitionistic fuzzy sets* defined on a non-empty set X as objects having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

Intuitionistic fuzzy sets

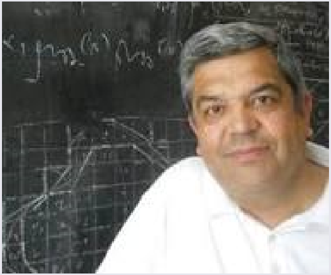


Figura: K. T. Atanassov

Intuitionistic fuzzy sets

For every two intuitionistic fuzzy sets A and B on X we define (see [?]):

(1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$, for all $x \in X$.

(2) $A^c = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$.

(3) $A \cap B =$
$\{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$.

(4) $A \cup B =$
$\{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$.

(5) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) \mid x \in X\}$.

(6) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) \mid x \in X\}$.

Group

Two Turkish Mathematicians who have worked on Intuitionistic Hyperstructures

Osman Kazanci

Bayram Ali Ersoy

Rough sets

The concept of rough set was originally proposed by Pawlak as a formal tool for modelling and processing in complete information in information systems. The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations.

Rough sets

Figura: Z. Pawlak

Rough sets

A key notion in Pawlak rough set model is an equivalence relation, i.e., a reflexive, symmetric and transitive relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set.

Rough sets

Suppose that U is a non-empty set. A partition or classification of U is a family $\mathcal{P}$ of non-empty subsets of U such that each element of U is contained in exactly one element of $\mathcal{P}$. Recall that an equivalence relation θ on a set U is a reflexive, symmetric, and transitive binary relation on U, i.e., for all $x, y, z \in U$, we have

$$\begin{array}{ll} x\theta x & \text{Reflexivity} \\ x\theta y \text{ implies } y\theta x & \text{Symmetry} \\ x\theta y \text{ and } y\theta z \text{ imply } x\theta z & \text{Transitivity} \end{array}$$

Rough sets

Each partition $\mathcal{P}$ induces an equivalence relation θ on U by setting

$$x\theta y \Longleftrightarrow x \text{ and } y \text{ are in the same class of } \mathcal{P}.$$

Conversely, each equivalence relation θ on U induces a partition $\mathcal{P}$ of U whose classes have the form $[x]_\theta = \{y \in U |\ x\theta y\}$.

The following notation will be used. Given a non-empty universe U, by $\mathcal{P}(U)$ we will denote a power-set on U. If θ is an equivalence relation on U then for every $x \in U$, $[x]_\theta$ stands for the equivalence class of θ with the represent x. For any $X \subseteq U$, we write X^c to denote the complement of X in U, that is the set $U \setminus X$.

Rough sets

Let U be a universe of objects and ρ be an equivalence relation on U. Given an arbitrary set $A \subseteq U$, a concept in U, it may be impossible to describe A precisely using the equivalence classes of ρ. That is, the available information is not sufficient to give a precise representation of A. In this case, one may characterize A by a pair of lower and upper approximations

$$\underline{app}(A) := \bigcup_{[a]_\rho \subseteq A} [a]_\rho \quad \text{and} \quad \overline{app}(A) := \bigcup_{[a]_\rho \cap A \neq \emptyset} [a]_\rho,$$

where $[a]_\rho = \{b \mid a \ \rho \ b\}$ is the equivalence class containing a.

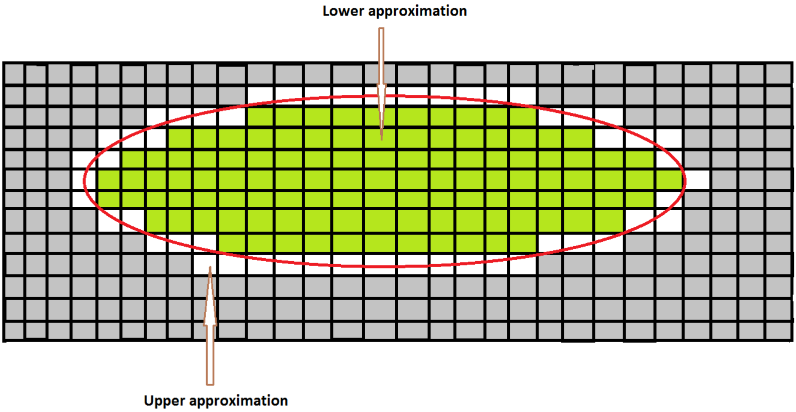
Rough sets

The lower approximation $\underline{app}(A)$ is the union of all the elementary sets which are subsets of A. The upper approximation $\overline{app}(A)$ is the union of all the elementary sets which have a non-empty intersection with A. An element in the lower approximation necessarily belongs to A, while an element in the upper approximation possibly belong to A. We can express lower and upper approximations as follows:

$$\underline{app}(A) = \{a \in U \mid [a]_\rho \subseteq A\} \text{ and}$$

$$\overline{app}(A) = \{a \in U \mid [a]_\rho \cap A \neq \emptyset\}.$$

Rough sets



Rough sets

The difference of upper and lower approximations is called *ρ-boundary region of A* and is denoted by $\widehat{app(A)}$.
A subset X of U is called definable if $\underline{app}(X) = \overline{app}(X)$. If $X \subseteq U$ is given by a predicate P and $x \in U$, then

(1) $x \in \underline{app}(X)$ means that x certainly has property P,

(2) $x \in \overline{app}(X)$ means that x possibly has property P,

(3) $x \in U \backslash \overline{app}(X)$ means that x definitely does not have property P.

Rough sets

Proposition

We have

(1) $\underline{app}(A) \subseteq A \subseteq \overline{app}(A)$;

(2) $\underline{app}(\emptyset) = \emptyset = \overline{app}(\emptyset)$;

(3) $\underline{app}(U) = U = \overline{app}(U)$;

(4) *If* $A \subseteq B$, *then* $\underline{app}(A) \subseteq \underline{app}(B)$ *and* $\overline{app}(A) \subseteq \overline{app}(B)$;

(5) $\underline{app}(\underline{app}(A)) = \underline{app}(A)$;

(6) $\overline{app}(\overline{app}(A)) = \overline{app}(A)$;

(7) $\overline{app}(\underline{app}(A)) = \underline{app}(A)$;

Rough sets

Proposition

(8) $\underline{app}(\overline{app}(A)) = \overline{app}(A)$;

(9) $\underline{app}(A) = (\overline{app}(A^C))^C$;

(10) $\overline{app}(A) = (\underline{app}(A^C))^C$;

(11) $\underline{app}(A \cap B) = \underline{app}(A) \cap \underline{app}(B)$;

(12) $\overline{app}(A \cap B) \subseteq \overline{app}(A) \cap \overline{app}(B)$;

(13) $\underline{app}(A \cup B) \supseteq \underline{app}(A) \cup \underline{app}(B)$;

(14) $\overline{app}(A \cup B) = \overline{app}(A) \cup \overline{app}(B)$.

Rough sets

A pair (U, ρ) where $U \neq \emptyset$ and ρ is an equivalence relation on U, is called an approximation space. For an approximation space (U, ρ), by a rough approximation in (U, ρ) we mean a mapping $app : \mathcal{P}(U) \to \mathcal{P}(U) \times \mathcal{P}(U)$ defined by for every $X \in \mathcal{P}(U)$,

$$app(X) = (\underline{app}(X), \overline{app}(X)).$$

Rough sets

The rough equality between sets is defined in the following way: for any $A, B \subseteq U$

$$A \approx B \;\Leftrightarrow\; \underline{app}(A) = \underline{app}(B) \text{ and } \overline{app}(A) = \overline{app}(B).$$

Obviously, $\approx$ is an equivalence relation on $P(U)$. Any equivalence class of the relation $\approx$ is called a rough set. We denote by

$$\mathcal{R}^0 = \{[X]_\approx \mid X \subseteq U\}$$

the family of all rough sets

Therefore, for a given approximation space (U, ρ), a pair $(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U)$ is a rough set in (U, ρ) if and only if $(A, B) = app(X)$ for some $X \in P(U)$.

Rough sets

Let $app(A) = (\underline{app}(A), \overline{app}(A))$ and $app(B) = (\underline{app}(B), \overline{app}(B))$ be any two rough sets in the approximation space (U, ρ). Then, we set the union, intersection, inclusion relation, complement, and set difference between rough sets as follows:

(1) $app(A) \sqcup app(B) := (\underline{app}(A) \cup \underline{app}(B), \overline{app}(A) \cup \overline{app}(B))$,

(2) $app(A) \sqcap app(B) := (\underline{app}(A) \cap \underline{app}(B), \overline{app}(A) \cap \overline{app}(B))$,

(3) $app(A) \sqsubseteq app(B) :\Leftrightarrow app(A) \cap app(B) = app(A)$.

Rough sets

When $app(A) \sqsubseteq app(B)$, we say that $app(A)$ is a rough subset of $app(B)$. Thus in the case of rough sets $app(A)$ and $app(B)$,

$$app(A) \sqsubseteq app(B) \text{ if and only if}$$
$$\underline{app}(A) \subseteq \underline{app}(B) \text{ and } \overline{app}(A) \subseteq \overline{app}(B).$$

This property of rough inclusion has all the properties of set inclusion. The rough complement of $app(A)$ denoted by $app^C(A)$ is defined by

$$app^C(A) := (U \backslash \overline{app}(A),\ U \backslash \underline{app}(A)).$$

Rough sets

Also, we can define $app(A) \backslash app(B)$ as follows:

$$\begin{aligned} app(A) \backslash app(B) &:= app(A) \cap app^C(B) \\ &= (\underline{app}(A) \backslash \overline{app}(B),\ \overline{app}(A) \backslash \underline{app}(B)). \end{aligned}$$

Rough sets

Theorem. Let (U, ρ) be an approximation space and $\mathcal{R}^0$ the family of rough sets. Then the algebra $(\mathcal{R}^0, \sqcup, \sqcap)$ is a complete distributive lattice.
The lattice $(\mathcal{R}^0, \sqcup, \sqcap)$ is bounded, where $0 = [\emptyset]_\approx$ is the least element and $1 = [U]_\approx$ is the greatest element.

Rough sets

For any $A \subseteq U$, a rough membership function is defined by

$$\mu_A(x) = \frac{|A \cap [x]_\rho|}{|[x]_\rho|}.$$

1) $\mu_A(x) = 1 \Longleftrightarrow x \in \underline{app}(A)$,
2) $\mu_A(x) = 0 \Longleftrightarrow x \in \underline{app}(A^c)$,
3) $0 < \mu_A(x) < 1 \Longleftrightarrow x \in \widehat{app(A)}$,
4) $\mu_A(x) = 1 - \mu_{A^c}$,
5) $\mu_{A\cup B} = \mu_A(x) + \mu_B(x) - \mu_{A\cap B}(x)$,
6) $\max\{\mu_A(x), \mu_B(x)\} \leq \mu_{A\cup B}(x) \leq \min\{1, \mu_A(x) + \mu_B(x)\}$,
7) $\mu_{A\cap B} \leq \min\{\mu_A(x), \mu_B(x)\}$.

Rough sets and hypergraph

Hypergraphs are like simple graphs, except that instead of having edges that only connect two vertices, their edges are sets of any number of vertices. This happens to mean that all graphs are just a subset of hypergraphs.

Definition. A *hypergraph* is a pair $\Gamma = (H, E)$, where H is a finite set of vertices and $E = \{E_1, \ldots, E_m\}$ is a set of hyperedges which are non-empty subsets of H such that

$$\bigcup_{i=1}^{m} E_i = H.$$

Rough sets and hypergraph

Connections between hypergraphs and hypergroups are studied by many authors. Corsini considered a hypergraph $\Gamma = (H, \{E_i\}_i)$ and defined a hyperoperation $\circ$ on H as follows:

$$\forall x, y \in H,\ x \circ y = E(x) \cup E(y),$$

where $E(x) = \bigcup_{x \in E_i} E_i$. The hypergroupoid $H_\Gamma = (H, \circ)$ is called a *hypergraph hypergroupoid* or a *h.g. hypergroupoid*. Corsini proved that

Rough sets and hypergraph

Theorem.[Corsini]. The hypergroupoid H_Γ has the following properties for each $(x, y) \in H^2$:

(1) $x \circ y = x \circ x \cup y \circ y$,

(2) $x \in x \circ x$,

(3) $y \in x \circ x \Longleftrightarrow x \in y \circ y$.

Rough sets and hypergraph

Also, he proved that:

Theorem.[Corsini].

(1) A hypergroupoid $(H, \circ)$ satisfying (1), (2) and (3) of previous theorem is a hypergroup if and only if the following condition is valid:

$$\forall (a, c) \in H^2,\ c \circ c \circ c \setminus c \circ c \subseteq a \circ a \circ a.$$

(2) A hypergroup $(H, \circ)$ satisfying (1), (2) and (3) of previous theorem is a join space.

An associative h.g hypergroupoid is called a *h.g. hypergroup*.

Rough sets and hypergraph

By using the concepts of lower and upper approximations, we define two new hyperoperations on the set of vertices of a given hypergraph.

Definition. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph, $E(x) = \bigcup_{x \in E_i} E_i$ and ρ be an equivalence relation on H. By using the equivalence relation ρ and the notions of lower and upper approximations, we define two hyperoperations on H as follows:

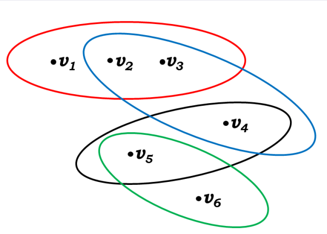
$$x \underline{\circ} y = \underline{app}(E(x)) \cup \underline{app}(E(y)) \cup \{x, y\},$$

$$x \overline{\circ} y = \overline{app}(E(x)) \cup \overline{app}(E(y)),$$

Rough sets and hypergraph

Note that $\overline{app}(E(x))$ is a non-empty set, for all $x \in H$. But it is possible to have the case $\underline{app}(E(x)) = \emptyset$, for some $x \in H$.

Example. Suppose that $H = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{E_1, E_2, E_3, E_4\}$, where $E_1 = \{v_1, v_2, v_3\}$, $E_2 = \{v_2, v_3, v_4\}$, $E_3 = \{v_5, v_6\}$ and $E_4 = \{v_4, v_5\}$.



Rough sets and hypergraph

Then, we have

$$\begin{aligned} E(v_1) &= E_1 = \{v_1, v_2, v_3\}, \\ E(v_2) &= E_1 \cup E_2 = \{v_1, v_2, v_3, v_4\}, \\ E(v_3) &= E_1 \cup E_2 = \{v_1, v_2, v_3, v_4\}, \\ E(v_4) &= E_2 \cup E_4 = \{v_2, v_3, v_4, v_5\}, \\ E(v_5) &= E_3 \cup E_4 = \{v_4, v_5, v_6\}, \\ E(v_6) &= E_3 = \{v_5, v_6\}. \end{aligned}$$

Let ρ be an equivalence relation on H that forms a partition on H as follows:

$$\{\{v_1, v_2\},\ \{v_3, v_5\},\ \{v_4\},\ \{v_6\}\}.$$

Rough sets and hypergraph

Hence

$$\underline{app}(E(v_1)) = \{v_1, v_2\},$$
$$\underline{app}(E(v_2)) = \{v_1, v_2, v_4\},$$
$$\underline{app}(E(v_3)) = \{v_1, v_2, v_4\},$$
$$\underline{app}(E(v_4)) = \{v_3, v_4, v_5\},$$
$$\underline{app}(E(v_5)) = \{v_4, v_6\},$$
$$\underline{app}(E(v_6)) = \{v_6\}.$$

Rough sets and hypergraph

Therefore, the lower hypergroupoid $\mathbb{H}_\rho = (H, \underline{\circ})$ associated to ρ is as follows:

$\underline{\circ}$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	$\{v_1, v_2\}$	$\{v_1, v_2, v_4\}$	$H \setminus \{v_5, v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_3\}$	$\{v_1, v_2, v_6\}$
v_2	$\{v_1, v_2, v_4\}$	$\{v_1, v_2\}$	$H \setminus \{v_5, v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_3\}$	$H \setminus \{v_3, v_5\}$
v_3	$H \setminus \{v_5, v_6\}$	$H \setminus \{v_5, v_6\}$	$H \setminus \{v_5, v_6\}$	$H \setminus \{v_6\}$	H	$H \setminus \{v_5\}$
v_4	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$\{v_3, v_4, v_5\}$	$H \setminus \{v_1, v_2\}$	$H \setminus \{v_1, v_2\}$
v_5	$H \setminus \{v_3\}$	$H \setminus \{v_3\}$	H	$H \setminus \{v_1, v_2\}$	$\{v_4, v_5, v_6\}$	$\{v_4, v_5, v_6\}$
v_6	$\{v_1, v_2, v_6\}$	$H \setminus \{v_3, v_5\}$	$H \setminus \{v_5\}$	$H \setminus \{v_1, v_2\}$	$\{v_4, v_5, v_6\}$	$\{v_6\}$

Rough sets and hypergraph

Similarly, we obtain

$$\begin{aligned}
\overline{app}(E(v_1)) &= \{v_1, v_2, v_3, v_5\}, \\
\overline{app}(E(v_2)) &= \{v_1, v_2, v_3, v_4, v_5\}, \\
\overline{app}(E(v_3)) &= \{v_1, v_2, v_3, v_4, v_5\}, \\
\overline{app}(E(v_4)) &= \{v_1, v_2, v_3, v_4, v_5\}, \\
\overline{app}(E(v_5)) &= \{v_3, v_4, v_5, v_6\}, \\
\overline{app}(E(v_6)) &= \{v_3, v_5, v_6\}.
\end{aligned}$$

Rough sets and hypergraph

Therefore, the upper hypergroupoid $\mathbb{H}^{\rho} = (H, \overline{\circ})$ associated to ρ is as follows:

$\overline{\circ}$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	$H \setminus \{v_4, v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	H	$H \setminus \{v_4\}$
v_2	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	H	H
v_3	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	H	H
v_4	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	$H \setminus \{v_6\}$	H	H
v_5	H	H	H	H	$H \setminus \{v_1, v_2\}$	$H \setminus \{v_1, v_2\}$
v_6	$H \setminus \{v_4\}$	H	H	H	$H \setminus \{v_1, v_2\}$	$H \setminus \{v_1, v_2, v_4\}$

Rough sets and hypergraph

Let $(H, \circ)$ and $(H, \star)$ be two hypergroupoids defined on the same set H.

(Vougiouklis.) The hyperoperation $\circ$ is called *smaller* than $\star$, or $\star$ *greater* than $\circ$ if there exists $f \in Aut(H)$ such that $x \circ y \subseteq f(x \star y)$, for all $x, y \in H$. In this case, we write $\circ \leqslant \star$ and we say that $(H, \star)$ contains $(H, \circ)$.

Corollary. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be an equivalence relation on H. If $\circ$ is the hyperoperation defined by Corsini, i.e., $x \circ y = E(x) \cup E(y)$, then

$$\underline{\circ} \leqslant \circ \leqslant \overline{\circ}.$$

Rough sets and hypergraph

Definition. We say an equivalence relation ρ on H is $\underline{\rho}$*-good* if

$$x \in \underline{app}E(y) \;\Leftrightarrow\; y \in \underline{app}E(x)$$

and we say ρ is $\overline{\rho}$*-good* if

$$x \in \overline{app}E(y) \;\Leftrightarrow\; y \in \overline{app}E(x).$$

Rough sets and hypergraph

Theorem. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be a $\underline{\rho}$-good equivalence relation on H. Then, $\mathbb{H}_\rho = (H, \underline{\circ})$ satisfies the following conditions:

(1) $x\underline{\circ}y = x\underline{\circ}x \cup y\underline{\circ}y$,

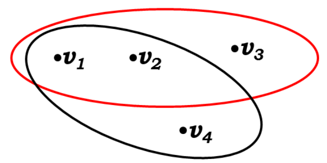
(2) $x \in x\underline{\circ}x$,

(3) $x \in y\underline{\circ}y \;\Leftrightarrow\; y \in x\underline{\circ}x$.

Corollary. $\mathbb{H}_\rho = (H, \underline{\circ})$ is a commutative H_v-group.
The hypergroup $\mathbb{H}_\rho$ is called the *lower $\underline{\rho}$-hypergroup induced by Γ*. Moreover, by Theorem **??**, $\mathbb{H}_\rho$ is a join space.

Rough sets and hypergraph

Example. Suppose that $H = \{v_1, v_2, v_3, v_4\}$ and $E = \{E_1, E_2\}$, where $E_1 = \{v_1, v_2, v_3\}$ and $E_2 = \{v_1, v_2, v_4\}$, see the following figure.



Rough sets and hypergraph

Then, we have

$$\begin{aligned} E(v_1) &= E_1 \cup E_2 = H, \\ E(v_2) &= E_1 \cup E_2 = H, \\ E(v_3) &= E_1 = \{v_1, v_2, v_3\} \\ E(v_4) &= E_2 = \{v_1, v_2, v_4\}. \end{aligned}$$

Let ρ be the following relation on H,

$$\begin{aligned} \rho = \quad & \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4), (v_1, v_2), \\ & (v_2, v_1), (v_3, v_4), (v_4, v_3)\}. \end{aligned}$$

Clearly, ρ is an equivalence relation.

Rough sets and hypergraph

Moreover, we have

$[v_1]_\rho \subseteq E(v_1)$	$[v_2]_\rho \subseteq E(v_1)$	$[v_3]_\rho \subseteq E(v_1)$	$[v_4]_\rho \subseteq E(v_1)$
$[v_1]_\rho \subseteq E(v_2)$	$[v_2]_\rho \subseteq E(v_2)$	$[v_3]_\rho \subseteq E(v_2)$	$[v_4]_\rho \subseteq E(v_2)$
$[v_1]_\rho \subseteq E(v_3)$	$[v_2]_\rho \subseteq E(v_3)$	$[v_3]_\rho \not\subseteq E(v_3)$	$[v_4]_\rho \not\subseteq E(v_3)$
$[v_1]_\rho \subseteq E(v_4)$	$[v_2]_\rho \subseteq E(v_4)$	$[v_3]_\rho \not\subseteq E(v_4)$	$[v_4]_\rho \not\subseteq E(v_4)$

This means that $x \in \underline{app}(E(y))$ if and only if $y \in \underline{app}(E(x))$.
So, ρ is a $\underline{\rho}$-good equivalence relation.

Rough sets and hypergraph

Now, we obtain

$$\underline{app}(E(v_1)) = H, \quad \underline{app}(E(v_2)) = H,$$
$$\underline{app}(E(v_3)) = \{v_1, v_2\}, \quad \underline{app}(E(v_4)) = \{v_1, v_2\}.$$

Therefore, the upper $\underline{\rho}$-hypergroup $\mathbb{H}_\rho = (H, \underline{\circ})$ induced by Γ is as follows:

$\underline{\circ}$	v_1	v_2	v_3	v_4
v_1	H	H	H	H
v_2	H	H	H	H
v_3	H	H	$\{v_1, v_2\}$	$\{v_1, v_2\}$
v_4	H	H	$\{v_1, v_2\}$	$\{v_1, v_2\}$

Rough sets and hypergraph

Farshi and Davvaz considered the special relation ρ on H as follows:

$$x \ \rho \ y \ \Leftrightarrow \ \{E_i \mid x \in E_i\} = \{E_j \mid y \in E_j\}.$$

Clearly, ρ is an equivalence relation. Moreover, we have the following two lemmas regarding to this special relation.

Lemma.

(1) ρ is a $\underline{\rho}$-good equivalence relation.

(2) ρ is a $\overline{\rho}$-good equivalence relation.

Lemma. For all $a \in H$, $E(x)$ is a definable set.

Rough sets and hypergraph

Theorem. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be a $\overline{\rho}$-good equivalence relation on H. Then, $(H, \overline{\circ})$ satisfies the following conditions:

(1) $x\overline{\circ}y = x\overline{\circ}x \cup y\overline{\circ}y$,

(2) $x \in x\overline{\circ}x$,

(3) $x \in y\overline{\circ}y \;\Leftrightarrow\; y \in x\overline{\circ}x$.

Corollary. $\mathbb{H}^\rho = (H, \overline{\circ})$ is a commutative H_v-group.

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Corollary. If the H_v-group $\mathbb{H}^\rho = (H, \overline{\circ})$ satisfies the following condition

$$x\overline{\circ}x\overline{\circ}x = x\overline{\circ}x, \text{ for all } x \in H,$$

then $\mathbb{H}_\rho = (H, \overline{\circ})$ is a hypergroup.

The hypergroup $\mathbb{H}^\rho$ is called the *upper $\overline{\rho}$-hypergroup induced by Γ*. Moreover, by Theorem **??**, $\mathbb{H}^\rho$ is a join space.

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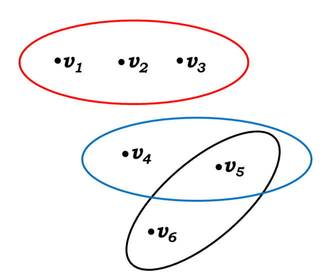
Example.

Suppose that $H = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{E_1, E_2, E_3\}$, where $E_1 = \{v_1, v_2, v_3\}$, $E_2 = \{v_4, v_5\}$ and $E_3 = \{v_5, v_6\}$, see the following figure.



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Then, we have

$$E(v_1) = E_1 = \{v_1, v_2, v_3\}, \qquad E(v_2) = E_1 = \{v_1, v_2, v_3\},$$
$$E(v_3) = E_1 = \{v_1, v_2, v_3\}, \qquad E(v_4) = E_2 = \{v_4, v_5\},$$
$$E(v_5) = E_2 \cup E_3 = \{v_4, v_5, v_6\},$$
$$E(v_6) = E_3 = \{v_5, v_6\}.$$

Let ρ be the following relation on H,

$$\begin{aligned}\rho = \ & \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4), (v_5, v_5), \\ & (v_6, v_6), (v_1, v_2), (v_2, v_1), (v_4, v_6), (v_6, v_4)\}.\end{aligned}$$

Clearly, ρ is an equivalence relation. Moreover, we have

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$[v_1]_\rho \cap E(v_1) \neq \emptyset$ $[v_1]_\rho \cap E(v_2) \neq \emptyset$ $[v_1]_\rho \cap E(v_3) \neq \emptyset$ $[v_1]_\rho \cap E(v_4) = \emptyset$ $[v_1]_\rho \cap E(v_5) = \emptyset$ $[v_1]_\rho \cap E(v_6) = \emptyset$	$[v_2]_\rho \cap E(v_1) \neq \emptyset$ $[v_2]_\rho \cap E(v_2) \neq \emptyset$ $[v_2]_\rho \cap E(v_3) \neq \emptyset$ $[v_2]_\rho \cap E(v_4) = \emptyset$ $[v_2]_\rho \cap E(v_5) = \emptyset$ $[v_2]_\rho \cap E(v_6) = \emptyset$	$[v_3]_\rho \cap E(v_1) \neq \emptyset$ $[v_3]_\rho \cap E(v_2) \neq \emptyset$ $[v_3]_\rho \cap E(v_3) \neq \emptyset$ $[v_3]_\rho \cap E(v_4) = \emptyset$ $[v_3]_\rho \cap E(v_5) = \emptyset$ $[v_3]_\rho \cap E(v_6) = \emptyset$
$[v_4]_\rho \cap E(v_1) = \emptyset$ $[v_4]_\rho \cap E(v_2) = \emptyset$ $[v_4]_\rho \cap E(v_3) = \emptyset$ $[v_4]_\rho \cap E(v_4) \neq \emptyset$ $[v_4]_\rho \cap E(v_5) \neq \emptyset$ $[v_4]_\rho \cap E(v_6) \neq \emptyset$	$[v_5]_\rho \cap E(v_1) = \emptyset$ $[v_5]_\rho \cap E(v_2) = \emptyset$ $[v_5]_\rho \cap E(v_3) = \emptyset$ $[v_5]_\rho \cap E(v_4) \neq \emptyset$ $[v_5]_\rho \cap E(v_5) \neq \emptyset$ $[v_5]_\rho \cap E(v_6) \neq \emptyset$	$[v_6]_\rho \cap E(v_1) = \emptyset$ $[v_6]_\rho \cap E(v_2) = \emptyset$ $[v_6]_\rho \cap E(v_3) = \emptyset$ $[v_6]_\rho \cap E(v_4) \neq \emptyset$ $[v_6]_\rho \cap E(v_5) \neq \emptyset$ $[v_6]_\rho \cap E(v_6) \neq \emptyset$

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Hence, we obtain

$$\overline{app}(E(v_1)) = \{v_1, v_2, v_3\},$$
$$\overline{app}(E(v_2)) = \{v_1, v_2, v_3\},$$
$$\overline{app}(E(v_3)) = \{v_1, v_2, v_3\},$$
$$\overline{app}(E(v_4)) = \{v_4, v_5, v_6\},$$
$$\overline{app}(E(v_5)) = \{v_4, v_5, v_6\},$$
$$\overline{app}(E(v_6)) = \{v_4, v_5, v_6\}.$$

Therefore, the upper $\overline{\rho}$-hypergroup $\mathbb{H}^{\rho} = (H, \overline{\circ})$ induced by Γ is as follows:

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$\overline{\circ}$	v_1	v_2	v_3	v_4	v_5
v_1	$\{v_1, v_2, v_3\}$	$\{v_1, v_2, v_3\}$	$\{v_1, v_2, v_3\}$	H	H
v_2	$\{v_1, v_2, v_3\}$	$\{v_1, v_2, v_3\}$	$\{v_1, v_2, v_3\}$	H	H
v_3	$\{v_1, v_2, v_3\}$	$\{v_1, v_2, v_3\}$	$\{v_1, v_2, v_3\}$	H	H
v_4	H	H	H	$\{v_4, v_5, v_6\}$	$\{v_4, v_5, v_6\}$
v_5	H	H	H	$\{v_4, v_5, v_6\}$	$\{v_4, v_5, v_6\}$
v_6	H	H	H	$\{v_4, v_5, v_6\}$	$\{v_4, v_5, v_6\}$

Note that in the above example, every element is an identity element.

Rough sets and hypergraph

Theorem. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be a $\overline{\rho}$-good equivalence relation on H. If $\mathbb{H}^\rho = (H, \overline{\circ})$ is the upper $\overline{\rho}$-hypergroup induced by Γ, then $\{x, y\} \subseteq x\overline{\circ}y$.

Theorem. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be a $\overline{\rho}$-good equivalence relation on H. If $\mathbb{H}^\rho = (H, \overline{\circ})$ is the upper $\overline{\circ}$-hypergroup induced by Γ, then $\beta^* = H^2$ where β^* is the fundamental relation on $\mathbb{H}^\rho$.

Rough sets and hypergraph

Theorem. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be a $\underline{\rho}$-good equivalence relation on H. If $\mathbb{H}_\rho = (H, \underline{\circ})$ is the lower $\underline{\circ}$-hypergroup induced by Γ, then $\beta^* = H^2$ where β^* is the fundamental relation on $\mathbb{H}^\rho$.

Let H be a hypergroup and A be a non-empty subset of H. We say that A is a *complete part* of H if for any nonzero natural number n and for all $a_1, \ldots, a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Longrightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

Rough sets and hypergraph

Theorem. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be a $\overline{\rho}$-good equivalence relation on H. If $\mathbb{H}^\rho = (H, \overline{\circ})$ is the upper $\overline{\circ}$-hypergroup induced by Γ, then a complete part of $\mathbb{H}^\rho$ is equal to H.

Theorem. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph and ρ be a $\underline{\rho}$-good equivalence relation on H. If $\mathbb{H}_\rho = (H, \underline{\circ})$ is the lower $\underline{\circ}$-hypergroup induced by Γ, then a complete part of $\mathbb{H}_\rho$ is equal to H.

Rough sets and hypergraph

A *hypergraph homomorphism* is a map from the vertex set of one hypergraph to another such that each hyperedge maps to one other hyperedge.

Let ρ_Γ and $\rho_{\Gamma'}$ be $\overline{\rho_\Gamma}$-good and $\overline{\rho_{\Gamma'}}$-good equivalence relations on hypergraphs Γ and Γ' respectively. A *ρ-isomorphism* from Γ to Γ' is a bijective homomorphism f from the vertex set of Γ to that of Γ' such that

$$x\rho_\Gamma y \Longleftrightarrow f(x)\rho_{\Gamma'} f(y).$$

We say that Γ and Γ' are ρ-isomorphic (written $\Gamma \cong_\rho \Gamma'$) if there is an isomorphism between them.

Rough sets and hypergraph

Lemma Let $\mathbb{H}^\rho = (H, \overline{\circ})$ and $\mathbb{H}'^\rho = (H', \overline{\star})$ be two upper $\overline{\rho}$-hypergroups induced by hypergraphs $\Gamma = (H, \{E_i\}_i)$ and $\Gamma' = (H', \{E'_j\}_j)$ respectively.

(1) If $f : \Gamma \to \Gamma'$ is a ρ-isomorphism, then $f(\overline{app}(E(x))) = \overline{app}(E'(f(x)))$.

(2) If $f : \mathbb{H}^\rho \to \mathbb{H}'^\rho$ is an isomorphism, then $f(\overline{app}(E(x))) = \overline{app}(E'(f(x)))$.

Rough sets and hypergraph

Theorem. Let $\mathbb{H}^{\rho} = (H, \overline{\circ})$ and $\mathbb{H}'^{\rho} = (H', \overline{\star})$ be two upper ρ-hypergroups induced by hypergraphs $\Gamma = (H, \{E_i\}_i)$ and $\Gamma' = (H', \{E'_j\}_j)$ respectively. If $\Gamma \cong \Gamma'$, then $\mathbb{H}^{\rho} \cong \mathbb{H}'^{\rho}$.

Lemma. Let $\mathbb{H}_{\rho} = (H, \underline{\circ})$ and $\mathbb{H}'_{\rho} = (H', \underline{\star})$ be two lower $\underline{\rho}$-hypergroups induced by hypergraphs $\Gamma = (H, \{E_i\}_i)$ and $\Gamma' = (H', \{E'_j\}_j)$ respectively.

(1) If $\Gamma \cong_{\rho} \Gamma'$, then $f(\underline{app}(E(x))) = \underline{app}(E'(f(x)))$.

(2) If $\mathbb{H}_{\rho} \cong \mathbb{H}'_{\rho}$, then $f(\underline{app}(E(x))) = \underline{app}(E'(f(x)))$.

Rough sets and hypergraph

Theorem. Let $\mathbb{H}_\rho = (H, \underline{\circ})$ and $\mathbb{H}'_\rho = (H', \underline{\star})$ be two lower $\underline{\rho}$-hypergroups induced by hypergraphs $\Gamma = (H, \{E_i\}_i)$ and $\Gamma' = (H', \{E'_j\}_j)$ respectively. If $\Gamma \cong_\rho \Gamma'$, then $\mathbb{H}_\rho \cong \mathbb{H}'_\rho$.

Main References

- P. Corsini, *Hypergraphs and hypergroups*, Algebra Universalis, 35 (1996) 548-555.
- P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, Tricesimo, 1993.
- Tita Khalis Maryati and Bijan Davvaz, *A novel connection between rough sets, hypergraphs and hypergroups*, Discrete Mathematics, Algorithms and Applications, 9(4) (2017) 1750044 (17 pages).

Mail References

- M. Farshi, B. Davvaz and S. Mirvakili, *Hypergraphs and hypergroups based on a special relation*, Communications in Algebra, 42 (2014), 3395-3406.
- M. Farshi, B. Davvaz and S. Mirvakili, *Degree hypergroupoids associated with hypergraphs*, Filomat, 28(1) (2014), 119-129.
- Z. Pawlak, *Rough sets*, Int. J. Inf. Comp. Sci., 11 (1982), 341-356.
- Z. Pawlak, *Rough Sets - Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishing, Dordrecht (1991).